25 $\mathcal{N} = 1$ Supergravity

25.1 More Superspace Notation

To allow general non-renormalizable interactions we can write

$$\int d^4\theta K(Q^\dagger, Q) \tag{25.1}$$

where K (the Kahler potential) is a real function. This is supersymmetric since the SUSY variation of the $\theta^2\theta^{\dagger 2}$ component of any superfield is a total derivative. We can also rewrite the vector multiplet as a real superfield:

$$V^a = \theta\bar{\sigma}^\mu\theta^\dagger A^a_\mu + i\theta^2\theta^\dagger\lambda^{a\dagger} - i\theta\theta^{\dagger 2}\lambda^a + \frac{1}{2}\theta^2\theta^{\dagger 2}D^a \tag{25.2}$$

so

$$V^a V^b = \frac{1}{2}\theta^2\theta^{\dagger 2}A^{a\mu}A^b_\mu \tag{25.3}$$

$$V^a V^b V^c = 0 \tag{25.4}$$

There is an extended gauge invariance that introduces additional auxiliary fields:

$$V^a \to V^a + \Lambda^a + \Lambda^{a\dagger} . \tag{25.5}$$

We can write our standard gauge interaction terms as

$$\int d^4\theta Q^\dagger e^V Q \tag{25.6}$$

where

$$V = 2gT^a V^a \tag{25.7}$$

25.2 Supergravity: On-Shell

Supergravity is a theory with local super-Poincare invariance. It is convenient to write the metric in terms of a vierbein (or tetrad):

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{M_{\text{Pl}}}e^m_\mu e^n_\nu \eta_{mn} \tag{25.8}$$

The vierbein e_μ^m corresponds to a helicity 2 particle (the graviton) $\mathcal{N} = 1$ SUSY requires a helicity 3/2 fermion $\psi_{\nu\alpha}$ (the gravitino). On shell (including the CPT conjugates) they each correspond to two degrees of freedom. Writing

$$e = |\det e_\mu^m| \tag{25.9}$$

the on-shell supergravity action is:

$$S = \frac{M_{\text{Pl}}}{2} \int d^4x\, eR + \frac{1}{2} \int d^4x\, \epsilon^{\mu\nu\rho\sigma} \overline{\psi}_\mu \gamma_5 \gamma_\nu D_\rho \psi_\sigma \tag{25.10}$$

where R is the curvature scalar, and D_ρ is a non-minimal covariant derivative.

25.3 Supergravity: Off-Shell

To understand the off-shell fields it is easier to first consider a theory with local superconformal-Poincare invariance. Recall the scale invariant Brans-Dicke gravity theory:

$$\mathcal{L}_{\text{BD}} = \frac{e}{2}\sigma^2 R + \frac{e}{12} \partial^\mu \sigma \partial_\mu \sigma \; . \tag{25.11}$$

If we treat σ as a spurion field and set

$$\sigma = M_{\text{Pl}} \tag{25.12}$$

the we just get Einstein gravity.

Now consider superconformal gravity. In addition to the "gauge" fields of supergravity (e_μ^m and $\psi_{\nu\alpha}$) we have a gauge field A_μ corresponding to the local R symmetry, and a "gauge" field b_μ corresponding to local conformal boosts. Counting degrees of freedom off-shell we have:

$$
\begin{array}{llll}
e_\mu^m : & 16 & -4 - 6 - 1 & = 5 \\
& & \text{general coord., local Lorentz, dilation} & \\
\psi_{\nu\alpha} : & 16 & -4 - 4 & = 8 \\
& & \text{SUSY, conformal SUSY} & \\
A_\mu : & 4 & -1 & = 3 \\
& & U(1)_R \text{ gauge} & \\
b_\mu : & 4 & -4 & = 0 \\
& & \text{conformal boost} &
\end{array}
$$

Since we have the same number of boson and fermion degrees of freedom, we don't need any auxiliary fields.

To get supergravity we need a spurion chiral superfield:

$$\Sigma = (\sigma, \chi, F_\Sigma) \tag{25.13}$$

We assign conformal weight 1 to Σ (x^μ and θ have conformal weight -1 and $-1/2$). Then the superconformal Lagrangian is

$$\mathcal{L}_{\text{sc}} = \frac{e\sigma^2}{2} R + e \int d^4\theta \Sigma^\dagger \Sigma + \mathcal{L}_{\text{gravitino}} \tag{25.14}$$

where all the derivatives are covariant in the four "gauge" fields.

Setting

$$\sigma = M_{\text{Pl}} \tag{25.15}$$
$$\chi = 0 \tag{25.16}$$
$$b_\mu = 0 \tag{25.17}$$

breaks local superconformal-Poincare invariance to local super-Poincare. The resulting Lagrangian is:

$$\mathcal{L}_{\text{sg}} = e \left[\frac{M_{\text{Pl}}^2}{2} R + F_\Sigma F_\Sigma^\dagger - \frac{2M_{\text{Pl}}^2}{9} A_\mu A^\mu \right] + \mathcal{L}_{\text{gravitino}} \tag{25.18}$$

so F_Σ and A_μ are auxiliary fields. Counting degrees of freedom we find:

$$
\begin{array}{llll}
e_\mu^m : & 16 & -4 - 6 & = 6 \\
 & & \text{general coord.}, \text{local Lorentz} \\
\psi_{\nu\alpha} : & 16 & -4 & = 12 \\
 & & \text{SUSY} \\
A_\mu : & 4 & & = 4 \\
F_\Sigma : & 2 & & = 2
\end{array}
$$

Given an arbitrary global SUSY theory:

$$\int d^4\theta K(Q^\dagger, e^V Q) + \int d^2\theta W(Q) - \int d^2\theta \frac{i\tau}{16\pi} W^\alpha W_\alpha + h.c. \tag{25.19}$$

We can make a local superconformal-Poincare invariant theory. We define conformal weight 0 fields and mass parameters by

$$Q' = \Sigma Q \tag{25.20}$$
$$m' = \Sigma m \tag{25.21}$$

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Dropping the primes we have

$$
\begin{aligned}
\mathcal{L} &= e\left[\int d^4\theta\, f(Q^\dagger, e^V Q)\frac{\Sigma^\dagger \Sigma}{M_{\text{Pl}}^2} + \int d^2\theta\, \frac{\Sigma^3}{M_{\text{Pl}}^3} W(Q) - \int d^2\theta\, \frac{i\tau}{16\pi} W^\alpha W_\alpha + h.c.\right] \\
&\quad - \frac{e}{6} f(q^\dagger, q)\sigma^*\sigma R + \mathcal{L}_{\text{aux}} + \mathcal{L}_{\text{gravitino}}
\end{aligned}
\tag{25.22}
$$

To make contact with the $M_{\text{Pl}} \to \infty$ limit we choose

$$
f = -3 M_{\text{Pl}}^2 e^{\frac{-K}{3 M_{\text{Pl}}^2}}
\tag{25.23}
$$

rescaling the metric

$$
e_\mu^m \to e^{\frac{-K}{12 M_{\text{Pl}}^2}} e_\mu^m
\tag{25.24}
$$

one finds for the bosonic piece of the Lagrangian

$$
\begin{aligned}
\mathcal{L}_{\text{B}} &= e\left[\frac{M_{\text{Pl}}^2}{2} R + K_j^i(q^\dagger, q) D^\mu q^{i\dagger} D_\mu q_j \right. \\
&\quad \left. - \mathcal{V}(q^\dagger, q) + \frac{i\tau}{16\pi}(F^2 + iF_{\mu\nu}\widetilde{F}^{\mu\nu}) + h.c.\right]
\end{aligned}
\tag{25.25}
$$

where the Kahler metric is

$$
K_j^i(q^\dagger, q) = \frac{\partial^2 K}{\partial q^{j\dagger}\partial q_i}
\tag{25.26}
$$

and

$$
\begin{aligned}
\mathcal{V}(q^\dagger, q) &= e^{\frac{K}{M_{\text{Pl}}^2}}\left[\left(K^{-1}\right)_i^j\left(W^i + \frac{WK^i}{M_{\text{Pl}}^2}\right)\left(W_j^* + \frac{W^*K_j}{M_{\text{Pl}}^2}\right) - \frac{3|W|^2}{M_{\text{Pl}}^2}\right] \\
&\quad + \frac{g^2}{2}\left(K^i T^a q_i\right)^2
\end{aligned}
\tag{25.27}
$$

So the energy density is not positive definite. The auxiliary (chiral) fields are given by

$$
F_i = -e^{\frac{K}{2M_{\text{Pl}}^2}}\left(K^{-1}\right)_i^j\left(W_j^* + \frac{W^*K_j}{M_{\text{Pl}}^2}\right)
\tag{25.28}
$$

From the fermion piece of the Lagrangian one notes that the covariant derivative $\sigma^\mu D_\mu$ of a fermion $\widetilde{q}$ contains

$$
\frac{1}{M_{\text{Pl}}}\sigma^\mu\overline{\psi}_{R\mu}\widetilde{q}
\tag{25.29}
$$

so the Kahler potential contains a term:

$$K^i_j \theta \frac{1}{M_{\mathrm{Pl}}} \sigma^\mu \overline{\psi}_{R\mu} \theta \widetilde{q} \theta^{\dagger 2} F^{j*} \ . \tag{25.30}$$

So we see that the gravitino gets a mass with (eats) the goldstino if there is a non-vanishing F^{j*}. The mass squared is given by

$$m^2_{3/2} = \frac{K^i_j F_i F^{*j}}{3M^2_{\mathrm{Pl}}} \ . \tag{25.31}$$

Taking a canonical Kahler potential

$$K = ZQ^{i\dagger}Q_i \tag{25.32}$$

and $M_{\mathrm{Pl}} \to \infty$ reproduces our global SUSY results.

References

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